Athena Transactions in Mathematical and Physical Sciences, Volume 1
Proceedings of the 1st International Symposium on Square Bamboos and the Geometree (ISSBG 2022), pp. 55–70
DOI: https://doi.org/10.55060/s.atmps.231115.006, ISSN (Online): 2949-9429
Proceedings home: https://www.athena-publishing.com/series/atmps/issbg-22

PROCEEDINGS ARTICLE

Laplace Transform Approximation of Nested Functions Using Bell's Polynomials

Paolo Emilio Ricci[1,*], Diego Caratelli[2,3], Sandra Pinelas[3,4]

[1] *UniNettuno International Telematic University, Rome, Italy*
[2] *Department of Electrical Engineering, Eindhoven University of Technology, Eindhoven, The Netherlands*
[3] *Department of Research and Development, The Antenna Company, Eindhoven, The Netherlands*
[4] *Department of Exact Sciences and Engineering, Portuguese Military Academy, Amadora, Portugal*

ABSTRACT

Bell's polynomials have been used in many different fields, ranging from number theory to operator theory. In this article we show a method to compute the Laplace Transform (LT) of nested analytic functions. To this aim, we provide a table of the first few values of the complete Bell's polynomials, which are then used to evaluate the LT of composed exponential functions. Furthermore, a code for approximating the LT of general analytic composed functions is created and presented. A graphical verification of the proposed technique is illustrated in the last section.

ARTICLE DATA

Article History
Received 10 January 2023
Revised 28 February 2023
Accepted 23 October 2023

Keywords
Laplace Transform
Bell's polynomials
Composed functions

1. INTRODUCTION

The common view that there is no formula for the Laplace Transform (LT) of composed analytic functions is disproved in this article, using Bell's polynomials [1], as in the case of the derivative of nested functions [2].

Bell's polynomials appear in very different fields, ranging from number theory [2,3,4,5,6] to operator theory [7], and from differential equations to integral transforms [8].

The importance of the LT is well known and it is not necessary to remind it here.

The second-order Bell polynomials $Y_n^{[2]}$representing the derivatives of nested functions of the type $f(g(h(t)))$ are then introduced, and two examples of LT of these functions are given. In Appendix II, a table of second-order Bell polynomials is reported, computed by the second author, using the Mathematica® program.

2. RECALLING THE BELL POLYNOMIALS

The Bell polynomials express the nth derivative of a composed function $\Phi(t) := f(g(t))$ in terms of the successive derivatives of the (sufficiently smooth) component functions $x = g(t)$ and $y = f(x)$. More precisely, if:

$$\Phi_m := D_t^m \Phi(t), \quad f_h := D_x^h f(x)\big|_{x=g(t)}, \quad g_k := D_t^k g(t)$$

then the nth derivative of $\Phi(t)$ is represented by:

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n)$$

where Y_n denotes the nth Bell polynomial.

*Corresponding author. Email: paoloemilioricci@gmail.com

The first few Bell polynomials are:

$$\begin{aligned} &Y_1(f_1, g_1) = f_1 g_1 \\ &Y_2(f_1, g_1; f_2, g_2) = f_1 g_2 + f_2 g_1^2 \\ &Y_3(f_1, g_1; f_2, g_2; f_3, g_3) = f_1 g_3 + f_2(3 g_2 g_1) + f_3 g_1^3 \end{aligned} \tag{1}$$

The Bell polynomials are given by:

$$\mathrm{Y}_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \textstyle\sum_{k=1}^{n} B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) f_k \tag{2}$$

The $B_{n,k}$ satisfy the recursion:

$$B_{n,k}(g_1, g_2, \dots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \dots, g_{n-h-k+1}) g_{h+1} \tag{3}$$

The $B_{n,k}$ functions for any $k = 1,2, \dots, n$ are polynomials homogeneous of degree k and *isobaric* of weight n (i.e. their monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ are such that $k_1 + 2k_2 + \cdots + nk_n = n$).

Therefore, we have the equations:

$$B_{n,k}(\alpha\beta g_1, \alpha\beta^2 g_2, \dots, \alpha\beta^{n-k+1} g_{n-k+1}) = \alpha^k \beta^n B_{n,k}(g_1, g_2, \dots, g_{n-k+1})$$

and:

$$Y_n(f_1, \beta g_1; f_2, \beta^2 g_2; \dots; f_n, \beta^n g_n) = \beta^n Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n)$$

An explicit expression for the Bell polynomials is given by the Faà di Bruno formula:

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \dots; f_n, g_n) = \sum_{p(n)} \frac{n!}{r_1! r_2! \dots r_n!} f_r \left(\frac{g_1}{1!}\right)^{r_1} \left(\frac{g_2}{2!}\right)^{r_2} \dots \left(\frac{g_n}{n!}\right)^{r_n} \tag{4}$$

where the sum runs over all partitions $p(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_n$ denotes the number of parts of the considered partition.

A proof of the Faà di Bruno formula can be found in [9]. The proof is based on the *umbral calculus* (see [10] and the references therein).

Remark 1: It should be noted that the possibility of constructing the Bell polynomials of index n by means of a recursion formula makes it possible to avoid their explicit form, which is expressed by means of the Faà di Bruno formula. This formula is not convenient from the computational point of view, because it makes use of partitions of the number n, and this number grows exponentially when n tends to infinity, as it is shown by the asymptotic behavior of the partition function by Hardy and Ramanujan [11]:

$$p(n) \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}}$$

3. RECALLING THE LAPLACE TRANSFORM

The Laplace Transform, a very useful tool in applied mathematics [12], writes:

$$\mathcal{L}(f) := \int_0^\infty \exp(-st) f(t) dt = F(s) \tag{5}$$

The Laplace operator converts a function of a real variable t (usually representing the time) to a function of a complex variable s (the complex frequency) and transforms differential into algebraic equations and convolution into multiplication.

It can be applied to functions belonging to $L^1_{loc}[0,+\infty)$ and it converges in each half plane Re $(s) > a$, where the *convergence abscissa* a, depends on the growth behavior of $f(t)$.

Remark 2: To avoid confusion, we want to stress that the purpose of this article is not to generalize the LT, but only to expand the table of transforms that are often used in applied mathematics problems, and which are reported in the book by Oberhettinger and Badii [13]. Actually, we give an approximation of the LT of composed analytic functions using elementary methods, namely the Taylor expansion and the Bell polynomials.

3.1. Main Properties and an Example

The Laplace transform method gives a rigorous approach to the operational technique introduced by Oliver Heaviside in 1893, in connection with his work in telegraphy.

This transformation is used to solve initial value problems for linear ordinary differential equations:

$$a_0 y(t) + a_1 y'(t) + \cdots + a_n y^{(n)}(t) = f(t)$$
$$y(0) = c_0, \quad y'(0) = c_1, \dots, y_{n-1}(0) = c_{n-1}$$

It can also be used for linear partial differential equations, and in particular in the case of the telegraphists' equation [14], which expresses the voltage v (or in equivalent form the current j) as a function of the constants that characterize the electrical circuit:

$$\frac{\partial^2 v}{\partial x^2} = \ell c \frac{\partial^2 v}{\partial t^2} + (rc + \ell g)\frac{\partial v}{\partial t} + rgv$$

where ℓ, r, c, g represent respectively the resistance, inductance, capacitance, conductance of the given circuit.

Note that this equation contains, as special cases, the vibrating string equation (when $r = g = 0$):

$$\frac{\partial^2 v}{\partial x^2} = \ell c \frac{\partial^2 v}{\partial t^2}$$

and the heat equation (when $\ell = g = 0$):

$$\frac{\partial^2 v}{\partial x^2} = rc \frac{\partial v}{\partial t}$$

so that the propagation of vibrations along a string and that of heat in a homogeneous medium can be seen as a particular case of the propagation of electricity along a wire.

The main rules are:

$$Linearity\ \mathcal{L}(Af + Bg) = AF(s) + BF(g)\ with\ A, B\ constants$$

Scaling property:

$$\mathcal{L}(f(at)) = \frac{1}{a} F\left(\frac{s}{a}\right) \qquad a > 0$$

Action on derivatives:

$$\mathcal{L}\left(\frac{df}{dt}\right) = sF(s) - f(0)$$

$$\mathcal{L}\left(\frac{d^2 f}{dt^2}\right) = s^2 F(s) - sf(0) - f'(0) \text{ etc.}$$

Convolution theorem:

$$\mathcal{L}(f) = F(s), \mathcal{L}(g) = G(s) \Rightarrow f * g := \mathcal{L}\left(\int_0^t f(t) g(t-\tau) e^{-\tau s} d\tau\right) = F(s)G(s)$$

Using these rules, and others derived from them and reported in suitable tables, the given equation in the time domain t is transformed into an equation in the frequency domain s, which is easier to solve, since the Laplace operator converts differential into algebraic equations and partial differential equations into ordinary ones.

After solving the problem in the frequency domain, the result is transformed back to the time domain, usually by using a table of inverse Laplace transforms or evaluating a Bromwich contour integral in the complex plane.

A simple example is the following.

Consider the harmonic oscillator problem:

$$y'' + \omega^2 y = f(t)$$

$$y(0) = a, \quad y'(0) = b$$

Multiplying by e^{-st} and integrating we find:

$$\int_0^\infty (y'' + \omega^2 y) e^{-st} dt = \int_0^\infty f(t) e^{-st} dt$$

$$\mathcal{L}(y'') + \omega^2 \mathcal{L}(y) = \mathcal{L}(f)$$

$$s^2 \mathcal{L}(y) - s y(0) - y'(0) + \omega^2 \mathcal{L}(y) = \mathcal{L}(f)$$

that is, using initial conditions:

$$(s^2 + \omega^2)\mathcal{L}(y) - as - b = \mathcal{L}(f)$$

$$\mathcal{L}(y) = \frac{\mathcal{L}(f)}{s^2 + \omega^2} + \frac{as + b}{s^2 + \omega^2}$$

Since:

$$\mathcal{L}(\sin\ \omega t) = \frac{\omega}{s^2 + \omega^2}, \quad \mathcal{L}(\cos\ \omega t) = \frac{s}{s^2 + \omega^2}$$

and recalling the convolution theorem we find:

$$\frac{\mathcal{L}(f)}{s^2 + \omega^2} = \frac{1}{\omega} \frac{\mathcal{L}(f)\omega}{s^2 + \omega^2} = \frac{1}{\omega} \mathcal{L}(f) \mathcal{L}(\sin \omega t) = \frac{1}{\omega} \mathcal{L}\left(\int_0^t f(\tau) \sin \omega (t - \tau)\, d\tau \right)$$

$$\mathcal{L}(y) = \mathcal{L}\left(\frac{1}{\omega} \int_0^t f(\tau) \sin \omega (t - \tau)\, d\tau + a \cos\ \omega t + \frac{b}{\omega} \sin\ \omega t \right)$$

so that inverting the Laplace transform we conclude that:

$$y(t) = \frac{1}{\omega} \int_0^t f(\tau) \sin \omega (t - \tau)\, d\tau + a \cos\ \omega t + \frac{b}{\omega} \sin\ \omega t$$

4. LAPLACE TRANSFORM OF COMPOSED FUNCTIONS

Consider a composed function $f(g(t))$ analytic in a neighborhood of the origin, so that it can be expressed by the Taylor's expansion:

$$f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_n = D_t^n [f(g(t))]_{t=0} \tag{6}$$

We have:

$$\begin{aligned} a_0 &= f\left(\mathring{g}_0\right) \\ a_n &= D_t^n [f(g(t))]_{t=0} = \sum_{k=1}^{n} B_{n,k}\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}\right) \mathring{f}_k \quad (n \geq 1) \end{aligned} \tag{7}$$

where:

$$\mathring{f}_k := D_x^k f(x)\big|_{x=g(0)}, \quad \mathring{g}_h := D_t^h g(t)\big|_{t=0} \tag{8}$$

Theorem 3. *Consider a composed function $f(g(t))$ which is analytic in a neighborhood of the origin and can be expressed by the Taylor's expansion in Eq. (6). For its LT the following equation holds:*

$$\int_0^{+\infty} f(g(t))e^{-ts}dt = \frac{f\left(\mathring{g}_0\right)}{s} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} B_{n,k}\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}\right)\mathring{f}_k\right)\frac{1}{s^{n+1}} \tag{9}$$

Proof. In fact, using the uniform convergence of Taylor's expansion, we can write:

$$\int_0^{+\infty} f(g(t))e^{-ts}dt = \frac{f\left(\mathring{g}_0\right)}{s} + \sum_{n=1}^{\infty}\int_0^{+\infty}\sum_{k=1}^{n} B_{n,k}\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}\right)\mathring{f}_k\frac{t^n}{n!}e^{-ts}dt =$$

$$\frac{f\left(\mathring{g}_0\right)}{s} + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} B_{n,k}\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}\right)\mathring{f}_k\right)\int_0^{+\infty}\frac{t^n}{n!}e^{-ts}dt$$

so that the conclusion follows by using the LT of powers.

5. THE CASE OF THE EXPONENTIAL FUNCTION

In the particular case when $f(x) = e^x$, that is considering the function $e^{g(t)}$ and assuming $g(0) = 0$, we then have the simpler form:

$$\sum_{k=1}^{n} B_{n,k}\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}\right)\mathring{f}_k = \sum_{k=1}^{n} B_{n,k}\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}\right) = B_n\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_n\right) \tag{10}$$

where the B_n are the complete Bell polynomials. It results $B_0(g_0) := f(g_0)$ and the first few values of B_n are:

$$\begin{aligned}
&B_1(g_1) = g_1\\
&B_2(g_1, g_2) = g_1^2 + g_2\\
&B_3(g_1, g_2, g_3) = g_1^3 + 3g_1g_2 + g_3\\
&B_4(g_1, g_2, g_3, g_4) = g_1^4 + 6g_1^2g_2 + 4g_1g_3 + 3g_2^2 + g_4
\end{aligned} \tag{11}$$

Further values are reported in Appendix I.

The complete Bell polynomials satisfy the identity:

$$B_{n+1}(g_1, \dots, g_{n+1}) = \sum_{k=0}^{n}\binom{n}{k}B_{n-k}(g_1, \dots, g_{n-k})g_{k+1} \tag{12}$$

In this case, the general Eq. (9) reduces to:

$$\int_0^{+\infty}\exp(g(t))\,e^{-ts}dt = \frac{\exp\left(\mathring{g}_0\right)}{s} + \sum_{n=1}^{\infty} B_n\left(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_n\right)\frac{1}{s^{n+1}} \tag{13}$$

In what follows we show the approximation of the LT of nested functions using the computer algebra program Mathematica®.

5.1. First Examples

We start considering the case of the LT of nested exponential functions:

- Let $f(x) = e^x$ and $g(t) = \sin t$. Then $g_1 = 1,\ g_2 = 0,\ g_3 = -1,\ g_4 = 0$, and in general $g_{2h} = 0,\ g_{2h+1} = (-1)^h,\ h = 1,2,3,\dots$

  According to Eq. (11) it results that:

  $$B_1(1) = 1,\ \ B_2(1,0) = 1,\ \ B_3(1,0,-1) = 0,\ \ B_4(1,0,-1,0) = -3,\ \ B_5(1,0,-1,0,1) = -8$$

  Then:

  $$\int_0^{+\infty} \exp(\sin t)\, e^{-ts} dt = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{3}{s^5} - \frac{8}{s^6} + O\left(\frac{1}{s^7}\right) \tag{14}$$

- Consider the Bessel function $g(t) := J_1(t)$ and the LT of the corresponding exponential function. We find:

  $$\int_0^{+\infty} \exp\big(J_1(t)\big) e^{-ts} dt = \frac{1}{s} + \frac{1}{2s^2} + \frac{1}{4s^3} - \frac{3}{4s^4} - \frac{11}{16s^5} - \frac{19}{32s^6} + \frac{91}{64s^7} + \frac{701}{128s^8} + \frac{953}{256s^9} - \frac{15245}{512s^{10}} + O\left(\frac{1}{s^{11}}\right) \tag{15}$$

- Let $g(t) = -\arctan(t)$. We find:

  $$\int_0^{+\infty} \exp(-\arctan(t))\, e^{-ts} dt = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} - \frac{7}{s^5} - \frac{5}{s^6} + \frac{145}{s^7} + \frac{5}{s^8} - \frac{6095}{s^9} - \frac{5815}{s^{10}} + O\left(\frac{1}{s^{11}}\right) \tag{16}$$

- Consider the complete elliptic integral of the second kind $g(t) := E(t)$ and the LT of the corresponding exponential function. We find:

  $$\int_0^{+\infty} \exp\big(E(t)\big)\, e^{-ts} dt = \frac{e^{\frac{\pi}{2}}}{s} - \frac{\pi}{8s^2} + \frac{\pi^2 - 3\pi}{64s^3} - \frac{\pi^3 - 9\pi^2 + 30\pi}{512s^4} + \frac{\pi^4 - 18\pi^3 + 147\pi^2 - 525\pi}{4096s^5} + O\left(\frac{1}{s^6}\right) \tag{17}$$

5.2. Graphical Display in Two Known Cases

5.2.1. Test Case #1

Considering the composed function $\cosh(\nu \operatorname{arcsinh}(t))$. It results [13]:

$$L(s) = \int_0^{+\infty} \cosh(\nu \operatorname{arcsinh}(t)) e^{-ts} dt = \frac{S_{1,\nu}(s)}{s} \qquad \Re s > 0 \tag{18}$$

where $S_{1,\nu}$ denotes a special case of the Lommel function $S_{\mu,\nu}$ [15]. Assuming $\nu = \pi$ and using our approximation we have found:

$$\int_0^{+\infty} \cosh[\pi \operatorname{arcsinh}(t)]\, e^{-ts} dt = \frac{1}{s} + \frac{\pi^2}{s^3} + \frac{\pi^2(\pi^2 - 4)}{s^5} + \frac{\pi^2(\pi^4 - 20\pi^2 + 64)}{s^7} + \frac{\pi^2(\pi^6 - 56\pi^4 + 784\pi^2 - 2304)}{s^9} + \frac{\pi^2(\pi^8 - 120\pi^6 + 4368\pi^4 - 52480\pi^2 + 147456)}{s^{11}} + O\left(\frac{1}{s^{13}}\right) \tag{19}$$

so that, by inverse Laplace transformation, one can readily conclude that:

$$\tilde{l}(t) \simeq \left(1 + \frac{\pi^2}{2!}t^2 + \frac{\pi^2(\pi^2-4)}{4!}t^4 + \frac{\pi^2(\pi^4-20\pi^2+64)}{6!}t^6 + \frac{\pi^2(\pi^6-56\pi^4+784\pi^2-2304)}{8!}t^8 + \frac{\pi^2(\pi^8-120\pi^6+4368\pi^4-52480\pi^2+147456)}{10!}t^{10}\right)H(t) \tag{20}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

The distributions of $L(s)$ and $\tilde{L}(s)$ along the cut sections $\omega = \Im s = 1$ and $\sigma = \Re s = 5$ are reported in Fig. 1 and Fig. 2, respectively. As it can be noticed, the agreement between the exact transform in Eq. (18) (for $\nu = \pi$) and the relevant approximation in Eq. (19) is very good especially as $s \to +\infty$. Conversely, the functions $l(t)$ and $\tilde{l}(t)$ tend to match for $t \to 0^+$as one would expect from theory (see Fig. 3).

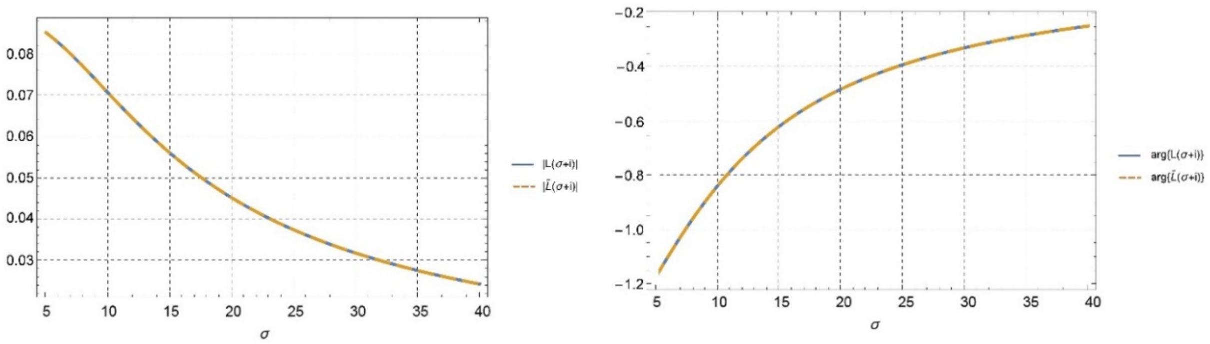
5.2.2. Test Case #2

Considering the composed function $J_\nu(a\sinh(t))$ with $\Re a > 0,\ \Re\nu > -1$, it results [13]:

$$L(s) = \int_0^{+\infty} J_\nu(a\sinh(t))e^{-ts}dt = J_{\frac{\nu+s}{2}}\left(\frac{a}{2}\right)K_{\frac{\nu-s}{2}}\left(\frac{a}{2}\right) \quad \Re s > -\frac{1}{2} \tag{21}$$

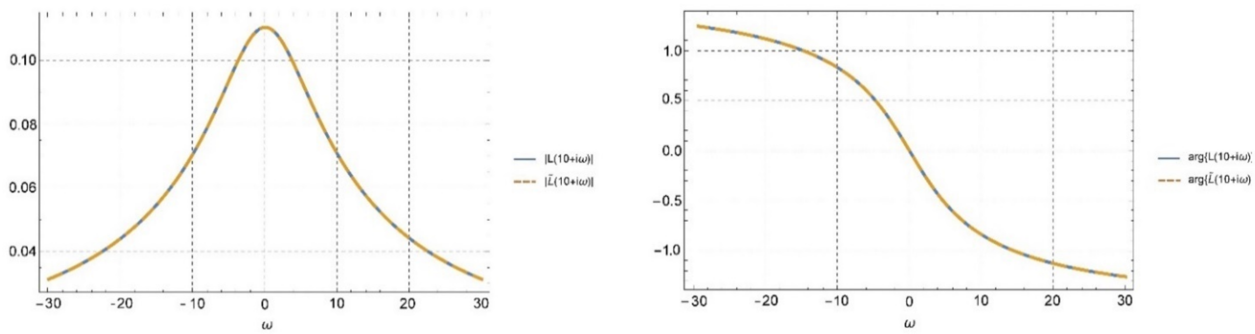
where J_ν and K_ν are Bessel functions. Assuming $\nu = 0$ and $a = 1$ we find the LT:

$$L(s) = \int_0^{+\infty} J_0(\sinh(t))e^{-ts}dt = J_{\frac{s}{2}}\left(\frac{1}{2}\right)K_{-\frac{s}{2}}\left(\frac{1}{2}\right) \quad \Re s > -\frac{1}{2} \tag{22}$$



(a) **(b)**

Figure 1. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \cosh[\pi\,\text{arcsinh}(t)]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\omega$ with $\omega = 1$.



(a) **(b)**

Figure 2. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \cosh[\pi\,\text{arcsinh}(t)]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\omega$ with $\sigma = 5$.

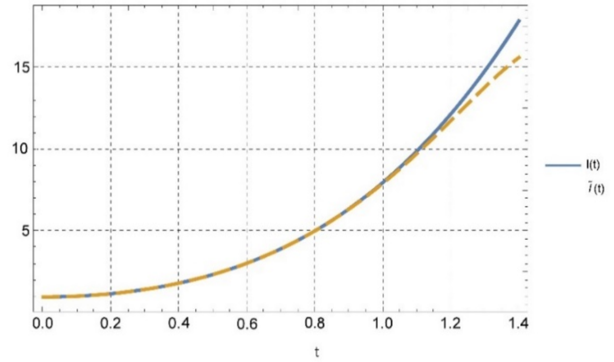


Figure 3. Distribution of $l(t) = \cosh[\pi \operatorname{arcsinh}(t)]$ and the relevant approximant $\tilde{l}(t)$.

Using our approximation, we have found:

$$L(s) \simeq \tilde{L}(s) = \int_0^{+\infty} J_0(\sinh(t))e^{-ts}dt = \frac{1}{s} - \frac{1}{2s^3} - \frac{13}{8s^5} - \frac{13}{16s^7} + \frac{9827}{128s^9} + \frac{309649}{256s^{11}} + O\left(\frac{1}{s^{13}}\right) \tag{23}$$

so that, by inverse Laplace transformation, one can readily conclude that:

$$\tilde{l}(t) \simeq \left(1 - \frac{1}{4}t^2 - \frac{13}{192}t^4 - \frac{13}{11520}t^6 + \frac{9827}{5160960}t^8 + \frac{309649}{928972800}t^{10}\right)H(t) \tag{24}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

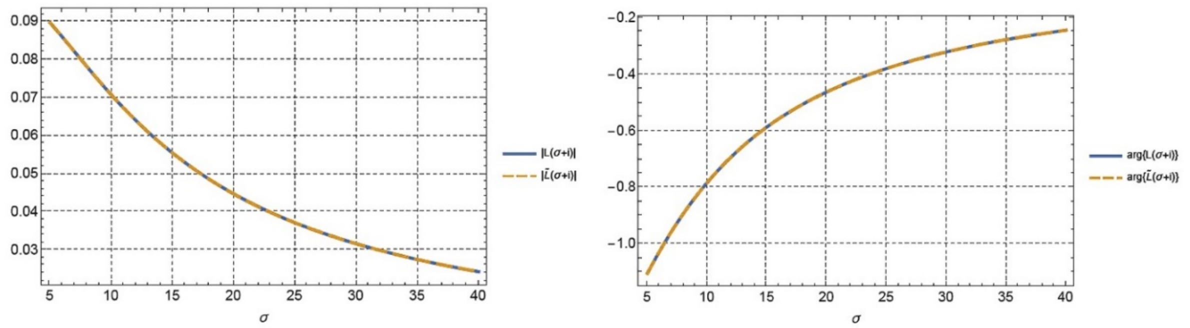
The distributions of $L(s)$ and $\tilde{L}(s)$ along the cut sections $\omega = \Im s = 1$ and $\sigma = \Re s = 5$ are reported in Fig. 4 and Fig. 5, respectively. As it can be noticed, the agreement between the exact transform in Eq. (22) and the relevant approximation in Eq. (23) is very good especially as $s \to +\infty$. Conversely, the functions $l(t)$ and $\tilde{l}(t)$ tend to match for $t \to 0^+$as one would expect from theory (see Fig. 6).

6. AN EXTENSION OF THE BELL POLYNOMIALS

We limit ourselves to the second-order Bell polynomials, $Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \dots; f_n, g_n, h_n)$, generated by the n-th derivative of the composed function $\Phi(t) := f(g(h(t)))$.

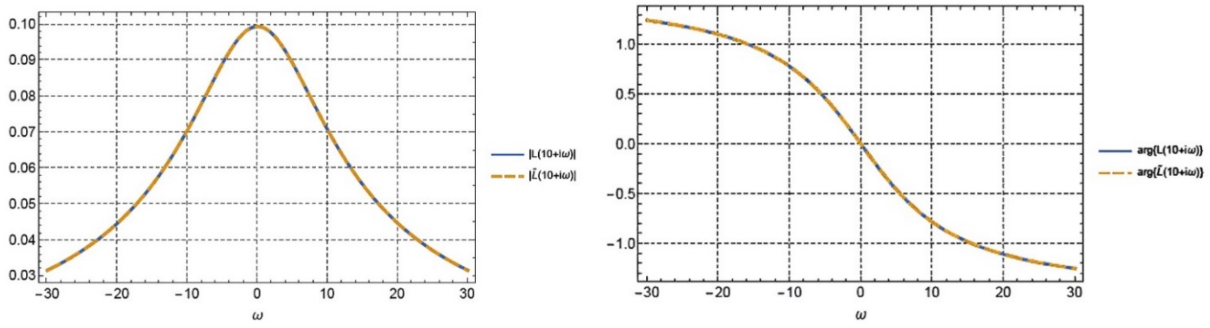
Consider the differentiable functions $x = h(t)$, $z = g(x)$ and $y = f(z)$, and suppose it is possible to use the chain rule for the n-th differentiation of the nested function $\Phi(t) := f(g(h(t)))$. We use the notations:

$$\Phi_j := D_t^j \Phi(t), \quad f_h := D_y^h f(y)\big|_{y=g(x)}, \quad g_k := D_x^k g(x)\big|_{x=h(t)}, \quad h_r := D_t^r h(t) \tag{25}$$



(a) **(b)**

Figure 4. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = J_0(\sinh(t))$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\omega$ with $\omega = 1$.



(a) **(b)**

Figure 5. Magnitude (a) and argument (b) of the Laplace transform of $l(t) = J_0(\sinh(t))$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\omega$ with $\sigma = 5$.

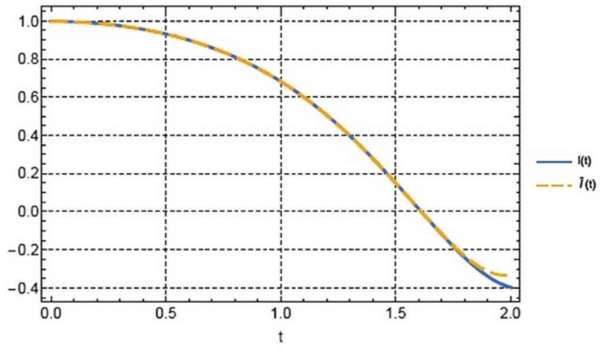


Figure 6. Distribution of $l(t) = J_0(\sinh(t))$ and the relevant approximant $\tilde{l}(t)$.

Then the n-th derivative can be represented as:

$$\Phi_n = Y_n^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; \dots; f_n, g_n, h_n)$$

where the $Y_n^{[2]}$ are the *second-order Bell polynomials* [16].

For example, one has:

$$Y_1^{[2]}(f_1, g_1, h_1) = f_1 g_1 h_1$$
$$Y_2^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2) = f_1 g_1 h_2 + f_1 g_2 h_1^2 + f_2 g_1^2 h_1^2$$
$$Y_3^{[2]}(f_1, g_1, h_1; f_2, g_2, h_2; f_3, g_3, h_3) = f_1 g_1 h_3 + f_1 g_3 h_1^3 + 3 f_1 g_2 h_1 h_2 + 3 f_2 g_1^2 h_1 h_2 + 3 f_2 g_1 g_2 h_1^3 + f_3 g_1^3 h_1^3$$

The connections with the ordinary Bell polynomials are expressed by the equation:

$$Y_n^{[2]}(f_1, g_1, h_1; \dots; f_n, g_n, h_n) = Y_n(f_1, Y_1(g_1, h_1); f_2, Y_2(g_1, h_1; g_2, h_2); \dots; f_n, Y_n(g_1, h_1; g_2, h_2; \dots; g_n, h_n))$$

Consequently, we deduce the theorem:

Theorem 4. *The following recurrence relation for the second-order Bell polynomials holds true:*

$$Y_0^{[2]} = f_1$$
$$Y_{n+1}^{[2]}(f_1, g_1, h_1; \dots; f_{n+1}, g_{n+1}, h_{n+1}) =$$
$$\sum_{k=0}^{n} \binom{n}{k} Y_{n-k}^{[2]}(f_2, g_1, h_1; f_3, g_2, h_2; \dots; f_{n-k+1}, g_{n-k}, h_{n-k}) Y_{k+1}(g_1, h_1; \dots; g_{k+1}, h_{k+1})$$

The first few second-order Bell polynomials are as follows:

$$\begin{aligned}
&Y_1^{[2]}([f,g,h]_1) = f_1g_1h_1\\
&Y_2^{[2]}([f,g,h]_2) = f_1g_1h_2 + f_1g_2h_1^2 + f_2g_1^2h_1^2\\
&Y_3^{[2]}([f,g,h]_3) = f_1g_1h_3 + f_1g_3h_1^3 + 3f_1g_2h_1h_2 + 3f_2g_1g_2h_1^3 + f_3g_1^3h_1^3\\
&Y_4^{[2]}([f,g,h]_4) = f_4g_1^4h_1^4 + 6f_3g_1^2g_2h_1^4 + 3f_2g_2^2h_1^4 + 4f_2g_1g_3h_1^4 + f_1g_4h_1^4 + 6f_3g_1^3h_1^2h_2 +\\
&\qquad 18f_2g_1g_2h_1^2h_2 + 6f_1g_3h_1^2h_2 + 3f_2g_1^2h_2^2 + 3f_1g_2h_2^2 + 4f_2g_1^2h_1h_3 + 4f_1g_2h_1h_3 + f_1g_1h_4
\end{aligned} \tag{26}$$

Further values are reported in Appendix II.

7. LAPLACE TRANSFORM OF NESTED FUNCTIONS

Let $f(g(h(t)))$ be a nested function analytic in a neighborhood of the origin, expressed by the Taylor's expansion:

$$f(g(h(t))) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}, \quad a_n = D_t^n \left[f\left(g\left(\left(h(t)\right)\right)\right]_{t=0} \tag{27}$$

It results:

$$\begin{aligned}
a_0 &= \overset{\circ}{f}_0 = f(g(h(0))\\
a_n &= D_t^n\left[f\left(g\left(\left(h(t)\right)\right)\right]_{t=0} = Y_n^{[2]}\left(\overset{\circ}{f}_1, \overset{\circ}{g}_1, \overset{\circ}{h}_1; \ldots; \overset{\circ}{f}_n, \overset{\circ}{g}_n, \overset{\circ}{h}_n\right) \quad (n \geq 1)
\end{aligned} \tag{28}$$

where:

$$\overset{\circ}{f}_h := D_x^h f(y)\big|_{y=g(0)}, \quad \overset{\circ}{g}_k := D_t^k g(x)\big|_{x=h(0)}, \quad \overset{\circ}{h}_r := D_t^r h(t)|_{t=0} \tag{29}$$

Theorem 5. *Consider a nested function $f(g(h(t)))$ which is analytic in a neighborhood of the origin and which can be represented by the Taylor's expansion in Eq. (27). For its LT the following expression holds:*

$$\begin{aligned}
\int_0^{+\infty} f(g((h(t)))e^{-ts}dt &= \frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{\infty} Y_n^{[2]}\left(\overset{\circ}{f}_1, \overset{\circ}{g}_1, \overset{\circ}{h}_1; \ldots; \overset{\circ}{f}_n, \overset{\circ}{g}_n, \overset{\circ}{h}\right)\frac{t^n}{n!}e^{-ts}dt =\\
&\frac{\overset{\circ}{f}_0}{s} + \sum_{n=1}^{\infty} Y_n^{[2]}\left(\overset{\circ}{f}_1, \overset{\circ}{g}_1, \overset{\circ}{h}_1; \ldots; \overset{\circ}{f}_n, \overset{\circ}{g}_n, \overset{\circ}{h}_n\right)\frac{1}{s^{n+1}}
\end{aligned} \tag{30}$$

Proof. It is a straightforward application of the definition of second-order Bell's polynomials.

7.1. Example #1

Assuming $f(x) = e^{x-1}$, $g(y) = \cos(y)$, $h(t) = \sin(t)$, it results:

$$\int_0^{+\infty} \exp[\cos(\sin(t)) - 1]\, e^{-ts}dt = \frac{1}{s} - \frac{1}{s^3} + \frac{8}{s^5} - \frac{127}{s^7} + \frac{3523}{s^9} - \frac{146964}{s^{11}} + O\left(\frac{1}{s^{13}}\right) \tag{31}$$

The corresponding inverse LT is approximated by:

$$\tilde{l}(t) \simeq \left(1 - \frac{1}{2}t^2 + \frac{1}{3}t^4 - \frac{127}{720}t^6 + \frac{3523}{40320}t^8 - \frac{12247}{302400}t^{10}\right)H(t) \tag{32}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

7.2. Example #2

Assuming $f(x) = \log\left(1+\frac{x}{2}\right),\ g(y) = \cosh(y) - 1,\ h(t) = \sin(t)$, it results:

$$\int_0^{+\infty} \log\left[1+\frac{\cosh(\sin(t))-1}{2}\right] e^{-ts} dt = \frac{1}{2s^3} - \frac{9}{4s^5} - \frac{27}{2s^7} + \frac{1169}{8s^9} - \frac{5869}{2s^{11}} + O\left(\frac{1}{s^{13}}\right) \tag{33}$$

The corresponding inverse LT is approximated by:

$$\tilde{l}(t) \simeq \left(\frac{1}{4}t^2 - \frac{3}{32}t^4 + \frac{3}{160}t^6 - \frac{167}{46080}t^8 + \frac{5869}{7257600}t^{10}\right) H(t) \tag{34}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

7.3. Example #3

Assuming $f(x) = e^x,\ g(y) = J_1(y),\ h(t) = \sin(t)$, it results:

$$\int_0^{+\infty} \exp\left[J_1(\sin(t))\right] e^{-ts} dt = \frac{1}{s} - \frac{1}{2s^2} + \frac{1}{4s^3} - \frac{3}{4s^4} - \frac{27}{16s^5} + \frac{77}{32s^6} + \frac{1227}{64s^7} + \frac{385}{128s^8} - \frac{82663}{256s^9} - \frac{439229}{512s^{10}} + \frac{6754489}{1024s^{11}} + O\left(\frac{1}{s^{12}}\right) \tag{35}$$

The corresponding inverse LT is approximated by:

$$\tilde{l}(t) \simeq \left(1 + \frac{1}{2}t + \frac{1}{8}t^2 - \frac{1}{8}t^3 - \frac{9}{128}t^4 + \frac{77}{3840}t^5 + \frac{409}{15360}t^6 + \frac{11}{18432}t^7 - \frac{11809}{1474560}t^8 - \frac{62747}{26542080}t^9 + \frac{964927}{530841600}t^{10}\right) H(t) \tag{36}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

7.4. Example #4

Assuming $f(x) = \arctan(x),\ g(y) = y^{\frac{1}{3}},\ h(t) = \cosh(t)$, it results:

$$\int_0^{+\infty} \arctan\left[(\cosh(t))^{\frac{1}{3}}\right] e^{-ts} dt = \frac{\pi}{4s} + \frac{1}{6s^3} - \frac{1}{3s^5} + \frac{43}{18s^7} - \frac{338}{9s^9} + \frac{18523}{18s^{11}} + O\left(\frac{1}{s^{13}}\right) \tag{37}$$

The corresponding inverse LT is approximated by:

$$\tilde{l}(t) \simeq \left(\frac{\pi}{4} + \frac{1}{12}t^2 - \frac{1}{72}t^4 + \frac{43}{12960}t^6 - \frac{169}{181440}t^8 + \frac{18523}{65318400}t^{10}\right) H(t) \tag{38}$$

with $H(\cdot)$ denoting the classical Heaviside distribution.

Remark 6: Note also that successive Bell polynomials are represented exclusively by sums, products and powers, avoiding operations that may generate numerical instability. The use of computers allows calculations to be performed stably and quickly, even though the number of products to be added increases rapidly with the number *n*. In our calculations it was possible to obtain a sufficient approximation by limiting ourselves to order *n = 10*.

8. CONCLUSION

We have presented a method for approximating the integral of analytic composed functions. Considering the Taylor expansion of the given function and representing their coefficients in terms of Bell's polynomials, the integral reduces to the computation of an approximating series, which obviously converges if the integral is convergent. This methodology has been applied to the LT of an analytic composed function, starting from the case of analytic nested exponential functions, based on the complete Bell polynomials, computed by using the program Mathematica®, and shown in Appendix I.

In the second part the LT of analytic nested functions is considered, and the second-order Bell's polynomials used in this approach are reported in Appendix II. We want to stress that, even if we dealt with a basic subject, we have not found in the literature any general method for approximating this type of LTs, a gap which, in our opinion, has been now filled up. A graphical verification of the proposed technique, performed in the case when both the analytical forms of the transform and anti-transform are known, proved the correctness of our results.

The method used in this article has also been applied in other cases such as:

- the LT of analytic composed functions of several variables [17,18];
- the LT of composed functions of two variables, making use of Bell's polynomials in two dimensions introduced and studied in a previous article [19,20];
- the sine and cosine Fourier transform of particular nested functions [21].

REFERENCES

[1] E.T. Bell. Exponential Polynomials. Annals of Mathematics, 1934, 35(2): 258–277. https://doi.org/10.2307/1968431

[2] J. Riordan. An Introduction to Combinatorial Analysis. Chichester: John Wiley & Sons, 1958.

[3] P. Natalini, P.E. Ricci. Bell Polynomials and Modified Bessel Functions of Half-Integral Order. Applied Mathematics and Computation, 2015, 268: 270–274. https://doi.org/10.1016/j.amc.2015.06.069

[4] R. Orozco López. Solution of the Differential Equation $y^{(k)} = e^{ay}$, Special Values of Bell Polynomials, and (k,a)-Autonomous Coefficients. Journal of Integer Sequences, 2021, 24(8): 21.8.6.

[5] F. Qi, D.-W. Niu, D. Lim, Y.-H. Yao. Special Values of the Bell Polynomials of the Second Kind for Some Sequences and Functions. Journal of Mathematical Analysis and Applications, 2020, 491(2): 124382. https://doi.org/10.1016/j.jmaa.2020.124382

[6] P.E. Ricci, P. Natalini. Bell Polynomials and 2nd Kind Hypergeometric Bernoulli Numbers. Seminar of I. Vekua Institute of Applied Mathematics, 2022, 48: 36–45.

[7] P.E. Ricci. Bell Polynomials and Generalized Laplace Transforms. Integral Transforms and Special Functions, 2022, 33(12): 966–977. https://doi.org/10.1080/10652469.2022.2059077

[8] P. Natalini, P.E. Ricci. Higher Order Bell Polynomials and the Relevant Integer Sequences. Applicable Analysis and Discrete Mathematics, 2017, 11(2): 327–339. https://doi.org/10.2298/AADM1702327N

[9] S.M. Roman. The Formula of Faà di Bruno. The American Mathematical Monthly, 1980, 87(10): 805–809. https://doi.org/10.1080/00029890.1980.11995156

[10] S.M. Roman, G.-C. Rota. The Umbral Calculus. Advances in Mathematics, 1978, 27(2): 95–188. https://doi.org/10.1016/0001-8708(78)90087-7

[11] G.H. Hardy, S. Ramanujan. Asymptotic Formulae in Combinatory Analysis. Proceedings of the London Mathematical Society, 1918, s2-17(1): 75–115. https://doi.org/10.1112/plms/s2-17.1.75

[12] A. Ghizzetti, A. Ossicini. Trasformate di Laplace e Calcolo Simbolico. Turin: Unione Tipografico-Editrice Torinese (UTET), 1971. (in Italian)

[13] F. Oberhettinger, L. Badii. Tables of Laplace Transforms. Heidelberg: Springer-Verlag, 1973. https://doi.org/10.1007/978-3-642-65645-3

[14] L. Amerio. Funzioni Analitiche e Trasformazione di Laplace. Milan: Tamburini, 1976. (in Italian)

[15] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi. Higher Transcendental Functions, Vol. II. New York: McGraw-Hill Book Company, 1953.

[16] P. Natalini, P.E. Ricci. An Extension of the Bell Polynomials. Computers & Mathematics With Applications, 2004, 47(4-5): 719–725. https://doi.org/10.1016/S0898-1221(04)90059-4

[17] A. Bernardini, P. Natalini, P.E. Ricci. Multidimensional Bell Polynomials of Higher Order. Computers & Mathematics With Applications, 2005, 50(10-12): 1697–1708. https://doi.org/10.1016/j.camwa.2005.05.008

[18] D. Caratelli, P.E. Ricci. Bell's Polynomials and Laplace Transform of Higher Order Nested Functions. Symmetry, 2022, 14(10): 2139. https://doi.org/10.3390/sym14102139

[19] D. Caratelli, R. Srivastava, P.E. Ricci. The Laplace Transform of Composed Functions and Bivariate Bell Polynomials. Axioms, 2022, 11(11): 591. https://doi.org/10.3390/axioms11110591

[20] S. Noschese, P.E. Ricci. Differentiation of Multivariable Composite Functions and Bell Polynomials. Journal of Computational Analysis and Applications, 2003, 5(3): 333–340. https://doi.org/10.1023/A:1023227705558

[21] D. Caratelli, P.E. Ricci. On a Set of Sine and Cosine Fourier Transforms of Nested Functions. Dolomites Research Notes on Approximation, 2022, 15(1): 11–19. https://doi.org/10.14658/PUPJ-DRNA-2022-1-2

APPENDIX I: TABLE OF COMPLETE BELL POLYNOMIALS

$$
\begin{aligned}
B_1 &= g_1, \\
B_2 &= g_1^2 + g_2, \\
B_3 &= g_1^3 + 3g_1g_2 + g_3, \\
B_4 &= g_1^4 + 6g_1^2g_2 + 4g_1g_3 + 3g_2^2 + g_4, \\
B_5 &= g_1^5 + 10g_1^3g_2 + 15g_1g_2^2 + 10g_1^2g_3 + 10g_2g_3 + 5g_1g_4 + g_5, \\
B_6 &= g_1^6 + 15g_1^4g_2 + 45g_1^2g_2^2 + 15g_2^3 + 20g_1^3g_3 + 60g_1g_2g_3 + 10g_3^2 + 15g_1^2g_4 + 15g_2g_4 + 6g_1g_5 + g_6, \\
B_7 &= g_1^7 + 21g_1^5g_2 + 105g_1^3g_2^2 + 105g_1g_2^3 + 35g_1^4g_3 + 210g_1^2g_2g_3 + 105g_2^2g_3 + 70g_1g_3^2 + 35g_1^3g_4 + \\
&\quad 105g_1g_2g_4 + 35g_3g_4 + 21g_1^2g_5 + 21g_2g_5 + 7g_1g_6 + g_7, \\
B_8 &= g_1^8 + 28g_1^6g_2 + 210g_1^4g_2^2 + 420g_1^2g_2^3 + 105g_2^4 + 56g_1^5g_3 + 560g_1^3g_2g_3 + 840g_1g_2^2g_3 + 280g_1^2g_3^2 + \\
&\quad 280g_2g_3^2 + 70g_1^4g_4 + 420g_1^2g_2g_4 + 210g_2^2g_4 + 280g_1g_3g_4 + 35g_4^2 + 56g_1^3g_5 + 168g_1g_2g_5 + \\
&\quad 56g_3g_5 + 28g_1^2g_6 + 28g_2g_6 + 8g_1g_7 + g_8, \\
B_9 &= g_1^9 + 36g_1^7g_2 + 378g_1^5g_2^2 + 1260g_1^3g_2^3 + 945g_1g_2^4 + 84g_1^6g_3 + 1260g_1^4g_2g_3 + 3780g_1^2g_2^2g_3 + \\
&\quad 1260g_2^3g_3 + 840g_1^3g_3^2 + 2520g_1g_2g_3^2 + 280g_3^3 + 126g_1^5g_4 + 1260g_1^3g_2g_4 + 1890g_1g_2^2g_4 + \\
&\quad 1260g_1^2g_3g_4 + 1260g_2g_3g_4 + 315g_1g_4^2 + 126g_1^4g_5 + 756g_1^2g_2g_5 + 378g_2^2g_5 + 504g_1g_3g_5 + \\
&\quad 126g_4g_5 + 84g_1^3g_6 + 252g_1g_2g_6 + 84g_3g_6 + 36g_1^2g_7 + 36g_2g_7 + 9g_1g_8 + g_9, \\
B_{10} &= g_1^{10} + 45g_1^8g_2 + 630g_1^6g_2^2 + 3150g_1^4g_2^3 + 4725g_1^2g_2^4 + 945g_2^5 + 120g_1^7g_3 + 2520g_1^5g_2g_3 + \\
&\quad 12600g_1^3g_2^2g_3 + 12600g_1g_2^3g_3 + 2100g_1^4g_3^2 + 12600g_1^2g_2g_3^2 + 6300g_2^2g_3^2 + 2800g_1g_3^3 + \\
&\quad 210g_1^6g_4 + 3150g_1^4g_2g_4 + 9450g_1^2g_2^2g_4 + 3150g_2^3g_4 + 4200g_1^3g_3g_4 + 12600g_1g_2g_3g_4 + \\
&\quad 2100g_3^2g_4 + 1575g_1^2g_4^2 + 1575g_2g_4^2 + 252g_1^5g_5 + 2520g_1^3g_2g_5 + 3780g_1g_2^2g_5 + 2520g_1^2g_3g_5 + \\
&\quad 2520g_2g_3g_5 + 1260g_1g_4g_5 + 126g_5^2 + 210g_1^4g_6 + 1260g_1^2g_2g_6 + 630g_2^2g_6 + 840g_1g_3g_6 + \\
&\quad 210g_4g_6 + 120g_1^3g_7 + 360g_1g_2g_7 + 120g_3g_7 + 45g_1^2g_8 + 45g_2g_8 + 10g_1g_9 + g_{10}\,.
\end{aligned}
$$

APPENDIX II: TABLE OF SECOND-ORDER BELL POLYNOMIALS

In[∘]:= $\texttt{Y[n\_] :=} \sum_{k=1}^{n} (\texttt{BellY[n, k, Table[}h_m\texttt{, \{m, 1, n - k + 1\}]]}\; g_k)$

In[∘]:= $\texttt{Y2[n\_] :=} \sum_{k=1}^{n} (\texttt{BellY[n, k, Table[Y[m], \{m, 1, n - k + 1\}]]}\; f_k)$

In[∘]:= `Y2[1] // FullSimplify // Expand`

Out[∘]= $f_1 g_1 h_1$

In[∘]:= `Y2[2] // FullSimplify // Expand`

Out[∘]= $f_2 g_1^2 h_1^2 + f_1 g_2 h_1^2 + f_1 g_1 h_2$

In[∘]:= `Y2[3] // FullSimplify // Expand`

Out[∘]= $f_3 g_1^3 h_1^3 + 3 f_2 g_1 g_2 h_1^3 + f_1 g_3 h_1^3 + 3 f_2 g_1^2 h_1 h_2 + 3 f_1 g_2 h_1 h_2 + f_1 g_1 h_3$

In[∘]:= `Y2[4] // FullSimplify // Expand`

Out[∘]= $f_4 g_1^4 h_1^4 + 6 f_3 g_1^2 g_2 h_1^4 + 3 f_2 g_2^2 h_1^4 + 4 f_2 g_1 g_3 h_1^4 + f_1 g_4 h_1^4 + 6 f_3 g_1^3 h_1^2 h_2 + 18 f_2 g_1 g_2 h_1^2 h_2 + 6 f_1 g_3 h_1^2 h_2 + 3 f_2 g_1^2 h_2^2 + 3 f_1 g_2 h_2^2 + 4 f_2 g_1^2 h_1 h_3 + 4 f_1 g_2 h_1 h_3 + f_1 g_1 h_4$

In[∘]:= `Y2[5] // FullSimplify // Expand`

Out[∘]= $f_5 g_1^5 h_1^5 + 10 f_4 g_1^3 g_2 h_1^5 + 15 f_3 g_1 g_2^2 h_1^5 + 10 f_3 g_1^2 g_3 h_1^5 + 10 f_2 g_2 g_3 h_1^5 + 5 f_2 g_1 g_4 h_1^5 + f_1 g_5 h_1^5 + 10 f_4 g_1^4 h_1^3 h_2 + 60 f_3 g_1^2 g_2 h_1^3 h_2 + 30 f_2 g_2^2 h_1^3 h_2 + 40 f_2 g_1 g_3 h_1^3 h_2 + 10 f_1 g_4 h_1^3 h_2 + 15 f_3 g_1^3 h_1 h_2^2 + 45 f_2 g_1 g_2 h_1 h_2^2 + 15 f_1 g_3 h_1 h_2^2 + 10 f_3 g_1^3 h_1^2 h_3 + 30 f_2 g_1 g_2 h_1^2 h_3 + 10 f_1 g_3 h_1^2 h_3 + 10 f_2 g_1^2 h_2 h_3 + 10 f_1 g_2 h_2 h_3 + 5 f_2 g_1^2 h_1 h_4 + 5 f_1 g_2 h_1 h_4 + f_1 g_1 h_5$

In[∘]:= `Y2[6] // FullSimplify // Expand`

Out[∘]= $f_6 g_1^6 h_1^6 + 15 f_5 g_1^4 g_2 h_1^6 + 45 f_4 g_1^2 g_2^2 h_1^6 + 15 f_3 g_2^3 h_1^6 + 20 f_4 g_1^3 g_3 h_1^6 + 60 f_3 g_1 g_2 g_3 h_1^6 + 10 f_2 g_3^2 h_1^6 + 15 f_3 g_1^2 g_4 h_1^6 + 15 f_2 g_2 g_4 h_1^6 + 6 f_2 g_1 g_5 h_1^6 + f_1 g_6 h_1^6 + 15 f_5 g_1^5 h_1^4 h_2 + 150 f_4 g_1^3 g_2 h_1^4 h_2 + 225 f_3 g_1 g_2^2 h_1^4 h_2 + 150 f_3 g_1^2 g_3 h_1^4 h_2 + 150 f_2 g_2 g_3 h_1^4 h_2 + 75 f_2 g_1 g_4 h_1^4 h_2 + 15 f_1 g_5 h_1^4 h_2 + 45 f_4 g_1^4 h_1^2 h_2^2 + 270 f_3 g_1^2 g_2 h_1^2 h_2^2 + 135 f_2 g_2^2 h_1^2 h_2^2 + 180 f_2 g_1 g_3 h_1^2 h_2^2 + 45 f_1 g_4 h_1^2 h_2^2 + 15 f_3 g_1^3 h_2^3 + 45 f_2 g_1 g_2 h_2^3 + 15 f_1 g_3 h_2^3 + 20 f_4 g_1^4 h_1^3 h_3 + 120 f_3 g_1^2 g_2 h_1^3 h_3 + 60 f_2 g_2^2 h_1^3 h_3 + 80 f_2 g_1 g_3 h_1^3 h_3 + 20 f_1 g_4 h_1^3 h_3 + 60 f_3 g_1^3 h_1 h_2 h_3 + 180 f_2 g_1 g_2 h_1 h_2 h_3 + 60 f_1 g_3 h_1 h_2 h_3 + 10 f_2 g_1^2 h_3^2 + 10 f_1 g_2 h_3^2 + 15 f_3 g_1^3 h_1^2 h_4 + 45 f_2 g_1 g_2 h_1^2 h_4 + 15 f_1 g_3 h_1^2 h_4 + 15 f_2 g_1^2 h_2 h_4 + 15 f_1 g_2 h_2 h_4 + 6 f_2 g_1^2 h_1 h_5 + 6 f_1 g_2 h_1 h_5 + f_1 g_1 h_6$

In[∘]:= `Y2[7] // FullSimplify // Expand`

Out[∘]=

$$
\begin{aligned}
&f_7 g_1^7 h_1^7 + 21 f_6 g_1^5 g_2 h_1^7 + 105 f_5 g_1^3 g_2^2 h_1^7 + 105 f_4 g_1 g_2^3 h_1^7 + 35 f_5 g_1^4 g_3 h_1^7 + 210 f_4 g_1^2 g_2 g_3 h_1^7 + \\
&105 f_3 g_2^2 g_3 h_1^7 + 70 f_3 g_1 g_3^2 h_1^7 + 35 f_4 g_1^3 g_4 h_1^7 + 105 f_3 g_1 g_2 g_4 h_1^7 + 35 f_2 g_3 g_4 h_1^7 + \\
&21 f_3 g_1^2 g_5 h_1^7 + 21 f_2 g_2 g_5 h_1^7 + 7 f_2 g_1 g_6 h_1^7 + f_1 g_7 h_1^7 + 21 f_6 g_1^6 h_1^5 h_2 + 315 f_5 g_1^4 g_2 h_1^5 h_2 + \\
&945 f_4 g_1^2 g_2^2 h_1^5 h_2 + 315 f_3 g_2^3 h_1^5 h_2 + 420 f_4 g_1^3 g_3 h_1^5 h_2 + 1260 f_3 g_1 g_2 g_3 h_1^5 h_2 + \\
&210 f_2 g_3^2 h_1^5 h_2 + 315 f_3 g_1^2 g_4 h_1^5 h_2 + 315 f_2 g_2 g_4 h_1^5 h_2 + 126 f_2 g_1 g_5 h_1^5 h_2 + 21 f_1 g_6 h_1^5 h_2 + \\
&105 f_5 g_1^5 h_1^3 h_2^2 + 1050 f_4 g_1^3 g_2 h_1^3 h_2^2 + 1575 f_3 g_1 g_2^2 h_1^3 h_2^2 + 1050 f_3 g_1^2 g_3 h_1^3 h_2^2 + \\
&1050 f_2 g_2 g_3 h_1^3 h_2^2 + 525 f_2 g_1 g_4 h_1^3 h_2^2 + 105 f_1 g_5 h_1^3 h_2^2 + 105 f_4 g_1^4 h_1 h_2^3 + 630 f_3 g_1^2 g_2 h_1 h_2^3 + \\
&315 f_2 g_2^2 h_1 h_2^3 + 420 f_2 g_1 g_3 h_1 h_2^3 + 105 f_1 g_4 h_1 h_2^3 + 35 f_5 g_1^5 h_1^4 h_3 + 350 f_4 g_1^3 g_2 h_1^4 h_3 + \\
&525 f_3 g_1 g_2^2 h_1^4 h_3 + 350 f_3 g_1^2 g_3 h_1^4 h_3 + 350 f_2 g_2 g_3 h_1^4 h_3 + 175 f_2 g_1 g_4 h_1^4 h_3 + 35 f_1 g_5 h_1^4 h_3 + \\
&210 f_4 g_1^4 h_1^2 h_2 h_3 + 1260 f_3 g_1^2 g_2 h_1^2 h_2 h_3 + 630 f_2 g_2^2 h_1^2 h_2 h_3 + 840 f_2 g_1 g_3 h_1^2 h_2 h_3 + \\
&210 f_1 g_4 h_1^2 h_2 h_3 + 105 f_3 g_1^3 h_2^2 h_3 + 315 f_2 g_1 g_2 h_2^2 h_3 + 105 f_1 g_3 h_2^2 h_3 + 70 f_3 g_1^3 h_1 h_3^2 + \\
&210 f_2 g_1 g_2 h_1 h_3^2 + 70 f_1 g_3 h_1 h_3^2 + 35 f_4 g_1^4 h_1^3 h_4 + 210 f_3 g_1^2 g_2 h_1^3 h_4 + 105 f_2 g_2^2 h_1^3 h_4 + \\
&140 f_2 g_1 g_3 h_1^3 h_4 + 35 f_1 g_4 h_1^3 h_4 + 105 f_3 g_1^3 h_1 h_2 h_4 + 315 f_2 g_1 g_2 h_1 h_2 h_4 + \\
&105 f_1 g_3 h_1 h_2 h_4 + 35 f_2 g_1^2 h_3 h_4 + 35 f_1 g_2 h_3 h_4 + 21 f_3 g_1^3 h_1^2 h_5 + 63 f_2 g_1 g_2 h_1^2 h_5 + \\
&21 f_1 g_3 h_1^2 h_5 + 21 f_2 g_1^2 h_2 h_5 + 21 f_1 g_2 h_2 h_5 + 7 f_2 g_1^2 h_1 h_6 + 7 f_1 g_2 h_1 h_6 + f_1 g_1 h_7
\end{aligned}
$$

In[∘]:= `Y2[8] // FullSimplify // Expand`

Out[∘]=

$$
\begin{aligned}
&f_8 g_1^8 h_1^8 + 28 f_7 g_1^6 g_2 h_1^8 + 210 f_6 g_1^4 g_2^2 h_1^8 + 420 f_5 g_1^2 g_2^3 h_1^8 + 105 f_4 g_2^4 h_1^8 + 56 f_6 g_1^5 g_3 h_1^8 + \\
&560 f_5 g_1^3 g_2 g_3 h_1^8 + 840 f_4 g_1 g_2^2 g_3 h_1^8 + 280 f_4 g_1^2 g_3^2 h_1^8 + 280 f_3 g_2 g_3^2 h_1^8 + 70 f_5 g_1^4 g_4 h_1^8 + \\
&420 f_4 g_1^2 g_2 g_4 h_1^8 + 210 f_3 g_2^2 g_4 h_1^8 + 280 f_3 g_1 g_3 g_4 h_1^8 + 35 f_2 g_4^2 h_1^8 + 56 f_4 g_1^3 g_5 h_1^8 + \\
&168 f_3 g_1 g_2 g_5 h_1^8 + 56 f_2 g_3 g_5 h_1^8 + 28 f_3 g_1^2 g_6 h_1^8 + 28 f_2 g_2 g_6 h_1^8 + 8 f_2 g_1 g_7 h_1^8 + f_1 g_8 h_1^8 + \\
&28 f_7 g_1^7 h_1^6 h_2 + 588 f_6 g_1^5 g_2 h_1^6 h_2 + 2940 f_5 g_1^3 g_2^2 h_1^6 h_2 + 2940 f_4 g_1 g_2^3 h_1^6 h_2 + 980 f_5 g_1^4 g_3 h_1^6 h_2 + \\
&5880 f_4 g_1^2 g_2 g_3 h_1^6 h_2 + 2940 f_3 g_2^2 g_3 h_1^6 h_2 + 1960 f_3 g_1 g_3^2 h_1^6 h_2 + 980 f_4 g_1^3 g_4 h_1^6 h_2 + \\
&2940 f_3 g_1 g_2 g_4 h_1^6 h_2 + 980 f_2 g_3 g_4 h_1^6 h_2 + 588 f_3 g_1^2 g_5 h_1^6 h_2 + 588 f_2 g_2 g_5 h_1^6 h_2 + \\
&196 f_2 g_1 g_6 h_1^6 h_2 + 28 f_1 g_7 h_1^6 h_2 + 210 f_6 g_1^6 h_1^4 h_2^2 + 3150 f_5 g_1^4 g_2 h_1^4 h_2^2 + 9450 f_4 g_1^2 g_2^2 h_1^4 h_2^2 + \\
&3150 f_3 g_2^3 h_1^4 h_2^2 + 4200 f_4 g_1^3 g_3 h_1^4 h_2^2 + 12\,600 f_3 g_1 g_2 g_3 h_1^4 h_2^2 + 2100 f_2 g_3^2 h_1^4 h_2^2 + \\
&3150 f_3 g_1^2 g_4 h_1^4 h_2^2 + 3150 f_2 g_2 g_4 h_1^4 h_2^2 + 1260 f_2 g_1 g_5 h_1^4 h_2^2 + 210 f_1 g_6 h_1^4 h_2^2 + 420 f_5 g_1^5 h_1^2 h_2^3 + \\
&4200 f_4 g_1^3 g_2 h_1^2 h_2^3 + 6300 f_3 g_1 g_2^2 h_1^2 h_2^3 + 4200 f_3 g_1^2 g_3 h_1^2 h_2^3 + 4200 f_2 g_2 g_3 h_1^2 h_2^3 + \\
&2100 f_2 g_1 g_4 h_1^2 h_2^3 + 420 f_1 g_5 h_1^2 h_2^3 + 105 f_4 g_1^4 h_2^4 + 630 f_3 g_1^2 g_2 h_2^4 + 315 f_2 g_2^2 h_2^4 + \\
&420 f_2 g_1 g_3 h_2^4 + 105 f_1 g_4 h_2^4 + 56 f_6 g_1^6 h_1^5 h_3 + 840 f_5 g_1^4 g_2 h_1^5 h_3 + 2520 f_4 g_1^2 g_2^2 h_1^5 h_3 + \\
&840 f_3 g_2^3 h_1^5 h_3 + 1120 f_4 g_1^3 g_3 h_1^5 h_3 + 3360 f_3 g_1 g_2 g_3 h_1^5 h_3 + 560 f_2 g_3^2 h_1^5 h_3 + \\
&840 f_3 g_1^2 g_4 h_1^5 h_3 + 840 f_2 g_2 g_4 h_1^5 h_3 + 336 f_2 g_1 g_5 h_1^5 h_3 + 56 f_1 g_6 h_1^5 h_3 + 560 f_5 g_1^5 h_1^3 h_2 h_3 + \\
&5600 f_4 g_1^3 g_2 h_1^3 h_2 h_3 + 8400 f_3 g_1 g_2^2 h_1^3 h_2 h_3 + 5600 f_3 g_1^2 g_3 h_1^3 h_2 h_3 + 5600 f_2 g_2 g_3 h_1^3 h_2 h_3 + \\
&2800 f_2 g_1 g_4 h_1^3 h_2 h_3 + 560 f_1 g_5 h_1^3 h_2 h_3 + 840 f_4 g_1^4 h_1 h_2^2 h_3 + 5040 f_3 g_1^2 g_2 h_1 h_2^2 h_3 + \\
&2520 f_2 g_2^2 h_1 h_2^2 h_3 + 3360 f_2 g_1 g_3 h_1 h_2^2 h_3 + 840 f_1 g_4 h_1 h_2^2 h_3 + 280 f_4 g_1^4 h_1^2 h_3^2 + \\
&1680 f_3 g_1^2 g_2 h_1^2 h_3^2 + 840 f_2 g_2^2 h_1^2 h_3^2 + 1120 f_2 g_1 g_3 h_1^2 h_3^2 + 280 f_1 g_4 h_1^2 h_3^2 + 280 f_3 g_1^3 h_2 h_3^2 + \\
&840 f_2 g_1 g_2 h_2 h_3^2 + 280 f_1 g_3 h_2 h_3^2 + 70 f_5 g_1^5 h_1^4 h_4 + 700 f_4 g_1^3 g_2 h_1^4 h_4 + 1050 f_3 g_1 g_2^2 h_1^4 h_4 + \\
&700 f_3 g_1^2 g_3 h_1^4 h_4 + 700 f_2 g_2 g_3 h_1^4 h_4 + 350 f_2 g_1 g_4 h_1^4 h_4 + 70 f_1 g_5 h_1^4 h_4 + 420 f_4 g_1^4 h_1^2 h_2 h_4 + \\
&2520 f_3 g_1^2 g_2 h_1^2 h_2 h_4 + 1260 f_2 g_2^2 h_1^2 h_2 h_4 + 1680 f_2 g_1 g_3 h_1^2 h_2 h_4 + 420 f_1 g_4 h_1^2 h_2 h_4 + \\
&210 f_3 g_1^3 h_2^2 h_4 + 630 f_2 g_1 g_2 h_2^2 h_4 + 210 f_1 g_3 h_2^2 h_4 + 280 f_3 g_1^3 h_1 h_3 h_4 + 840 f_2 g_1 g_2 h_1 h_3 h_4 + \\
&280 f_1 g_3 h_1 h_3 h_4 + 35 f_2 g_1^2 h_4^2 + 35 f_1 g_2 h_4^2 + 56 f_4 g_1^4 h_1^3 h_5 + 336 f_3 g_1^2 g_2 h_1^3 h_5 + \\
&168 f_2 g_2^2 h_1^3 h_5 + 224 f_2 g_1 g_3 h_1^3 h_5 + 56 f_1 g_4 h_1^3 h_5 + 168 f_3 g_1^3 h_1 h_2 h_5 + 504 f_2 g_1 g_2 h_1 h_2 h_5 + \\
&168 f_1 g_3 h_1 h_2 h_5 + 56 f_2 g_1^2 h_3 h_5 + 56 f_1 g_2 h_3 h_5 + 28 f_3 g_1^3 h_1^2 h_6 + 84 f_2 g_1 g_2 h_1^2 h_6 + \\
&28 f_1 g_3 h_1^2 h_6 + 28 f_2 g_1^2 h_2 h_6 + 28 f_1 g_2 h_2 h_6 + 8 f_2 g_1^2 h_1 h_7 + 8 f_1 g_2 h_1 h_7 + f_1 g_1 h_8
\end{aligned}
$$

```
In[∘]:= Y2[9] // FullSimplify // Expand
```

Out[∘]=

$$
\begin{aligned}
& f_9 g_1^9 h_1^9 + 36 f_8 g_1^7 g_2 h_1^9 + 378 f_7 g_1^5 g_2^2 h_1^9 + 1260 f_6 g_1^3 g_2^3 h_1^9 + 945 f_5 g_1 g_2^4 h_1^9 + \\
& 84 f_7 g_1^6 g_3 h_1^9 + 1260 f_6 g_1^4 g_2 g_3 h_1^9 + 3780 f_5 g_1^2 g_2^2 g_3 h_1^9 + 1260 f_4 g_2^3 g_3 h_1^9 + \\
& 840 f_5 g_1^3 g_3^2 h_1^9 + 2520 f_4 g_1 g_2 g_3^2 h_1^9 + 280 f_3 g_3^3 h_1^9 + 126 f_6 g_1^5 g_4 h_1^9 + 1260 f_5 g_1^3 g_2 g_4 h_1^9 + \\
& 1890 f_4 g_1 g_2^2 g_4 h_1^9 + 1260 f_4 g_1^2 g_3 g_4 h_1^9 + 1260 f_3 g_2 g_3 g_4 h_1^9 + 315 f_3 g_1 g_4^2 h_1^9 + \\
& 126 f_5 g_1^4 g_5 h_1^9 + 756 f_4 g_1^2 g_2 g_5 h_1^9 + 378 f_3 g_2^2 g_5 h_1^9 + 504 f_3 g_1 g_3 g_5 h_1^9 + 126 f_2 g_4 g_5 h_1^9 + \\
& 84 f_4 g_1^3 g_6 h_1^9 + 252 f_3 g_1 g_2 g_6 h_1^9 + 84 f_2 g_3 g_6 h_1^9 + 36 f_3 g_1^2 g_7 h_1^9 + 36 f_2 g_2 g_7 h_1^9 + \\
& 9 f_2 g_1 g_8 h_1^9 + f_1 g_9 h_1^9 + 36 f_8 g_1^8 h_1^7 h_2 + 1008 f_7 g_1^6 g_2 h_1^7 h_2 + 7560 f_6 g_1^4 g_2^2 h_1^7 h_2 + \\
& 15\,120 f_5 g_1^2 g_2^3 h_1^7 h_2 + 3780 f_4 g_2^4 h_1^7 h_2 + 2016 f_6 g_1^5 g_3 h_1^7 h_2 + 20\,160 f_5 g_1^3 g_2 g_3 h_1^7 h_2 + \\
& 30\,240 f_4 g_1 g_2^2 g_3 h_1^7 h_2 + 10\,080 f_4 g_1^2 g_3^2 h_1^7 h_2 + 10\,080 f_3 g_2 g_3^2 h_1^7 h_2 + 2520 f_5 g_1^4 g_4 h_1^7 h_2 + \\
& 15\,120 f_4 g_1^2 g_2 g_4 h_1^7 h_2 + 7560 f_3 g_2^2 g_4 h_1^7 h_2 + 10\,080 f_3 g_1 g_3 g_4 h_1^7 h_2 + 1260 f_2 g_4^2 h_1^7 h_2 + \\
& 2016 f_4 g_1^3 g_5 h_1^7 h_2 + 6048 f_3 g_1 g_2 g_5 h_1^7 h_2 + 2016 f_2 g_3 g_5 h_1^7 h_2 + 1008 f_3 g_1^2 g_6 h_1^7 h_2 + \\
& 1008 f_2 g_2 g_6 h_1^7 h_2 + 288 f_2 g_1 g_7 h_1^7 h_2 + 36 f_1 g_8 h_1^7 h_2 + 378 f_7 g_1^7 h_1^5 h_2^2 + 7938 f_6 g_1^5 g_2 h_1^5 h_2^2 + \\
& 39\,690 f_5 g_1^3 g_2^2 h_1^5 h_2^2 + 39\,690 f_4 g_1 g_2^3 h_1^5 h_2^2 + 13\,230 f_5 g_1^4 g_3 h_1^5 h_2^2 + 79\,380 f_4 g_1^2 g_2 g_3 h_1^5 h_2^2 + \\
& 39\,690 f_3 g_2^2 g_3 h_1^5 h_2^2 + 26\,460 f_3 g_1 g_3^2 h_1^5 h_2^2 + 13\,230 f_4 g_1^3 g_4 h_1^5 h_2^2 + 39\,690 f_3 g_1 g_2 g_4 h_1^5 h_2^2 + \\
& 13\,230 f_2 g_3 g_4 h_1^5 h_2^2 + 7938 f_3 g_1^2 g_5 h_1^5 h_2^2 + 7938 f_2 g_2 g_5 h_1^5 h_2^2 + 2646 f_2 g_1 g_6 h_1^5 h_2^2 + \\
& 378 f_1 g_7 h_1^5 h_2^2 + 1260 f_6 g_1^6 h_1^3 h_2^3 + 18\,900 f_5 g_1^4 g_2 h_1^3 h_2^3 + 56\,700 f_4 g_1^2 g_2^2 h_1^3 h_2^3 + \\
& 18\,900 f_3 g_2^3 h_1^3 h_2^3 + 25\,200 f_4 g_1^3 g_3 h_1^3 h_2^3 + 75\,600 f_3 g_1 g_2 g_3 h_1^3 h_2^3 + 12\,600 f_2 g_3^2 h_1^3 h_2^3 + \\
& 18\,900 f_3 g_1^2 g_4 h_1^3 h_2^3 + 18\,900 f_2 g_2 g_4 h_1^3 h_2^3 + 7560 f_2 g_1 g_5 h_1^3 h_2^3 + 1260 f_1 g_6 h_1^3 h_2^3 + \\
& 945 f_5 g_1^5 h_1 h_2^4 + 9450 f_4 g_1^3 g_2 h_1 h_2^4 + 14\,175 f_3 g_1 g_2^2 h_1 h_2^4 + 9450 f_3 g_1^2 g_3 h_1 h_2^4 + \\
& 9450 f_2 g_2 g_3 h_1 h_2^4 + 4725 f_2 g_1 g_4 h_1 h_2^4 + 945 f_1 g_5 h_1 h_2^4 + 84 f_7 g_1^7 h_1^6 h_3 + 1764 f_6 g_1^5 g_2 h_1^6 h_3 + \\
& 8820 f_5 g_1^3 g_2^2 h_1^6 h_3 + 8820 f_4 g_1 g_2^3 h_1^6 h_3 + 2940 f_5 g_1^4 g_3 h_1^6 h_3 + 17\,640 f_4 g_1^2 g_2 g_3 h_1^6 h_3 + \\
& 8820 f_3 g_2^2 g_3 h_1^6 h_3 + 5880 f_3 g_1 g_3^2 h_1^6 h_3 + 2940 f_4 g_1^3 g_4 h_1^6 h_3 + 8820 f_3 g_1 g_2 g_4 h_1^6 h_3 + \\
& 2940 f_2 g_3 g_4 h_1^6 h_3 + 1764 f_3 g_1^2 g_5 h_1^6 h_3 + 1764 f_2 g_2 g_5 h_1^6 h_3 + 588 f_2 g_1 g_6 h_1^6 h_3 + \\
& 84 f_1 g_7 h_1^6 h_3 + 1260 f_6 g_1^6 h_1^4 h_2 h_3 + 18\,900 f_5 g_1^4 g_2 h_1^4 h_2 h_3 + 56\,700 f_4 g_1^2 g_2^2 h_1^4 h_2 h_3 + \\
& 18\,900 f_3 g_2^3 h_1^4 h_2 h_3 + 25\,200 f_4 g_1^3 g_3 h_1^4 h_2 h_3 + 75\,600 f_3 g_1 g_2 g_3 h_1^4 h_2 h_3 + \\
& 12\,600 f_2 g_3^2 h_1^4 h_2 h_3 + 18\,900 f_3 g_1^2 g_4 h_1^4 h_2 h_3 + 18\,900 f_2 g_2 g_4 h_1^4 h_2 h_3 + 7560 f_2 g_1 g_5 h_1^4 h_2 h_3 + \\
& 1260 f_1 g_6 h_1^4 h_2 h_3 + 3780 f_5 g_1^5 h_1^2 h_2^2 h_3 + 37\,800 f_4 g_1^3 g_2 h_1^2 h_2^2 h_3 + 56\,700 f_3 g_1 g_2^2 h_1^2 h_2^2 h_3 + \\
& 37\,800 f_3 g_1^2 g_3 h_1^2 h_2^2 h_3 + 37\,800 f_2 g_2 g_3 h_1^2 h_2^2 h_3 + 18\,900 f_2 g_1 g_4 h_1^2 h_2^2 h_3 + 3780 f_1 g_5 h_1^2 h_2^2 h_3 + \\
& 1260 f_4 g_1^4 h_2^3 h_3 + 7560 f_3 g_1^2 g_2 h_2^3 h_3 + 3780 f_2 g_2^2 h_2^3 h_3 + 5040 f_2 g_1 g_3 h_2^3 h_3 + 1260 f_1 g_4 h_2^3 h_3 + \\
& 840 f_5 g_1^5 h_1^3 h_3^2 + 8400 f_4 g_1^3 g_2 h_1^3 h_3^2 + 12\,600 f_3 g_1 g_2^2 h_1^3 h_3^2 + 8400 f_3 g_1^2 g_3 h_1^3 h_3^2 + \\
& 8400 f_2 g_2 g_3 h_1^3 h_3^2 + 4200 f_2 g_1 g_4 h_1^3 h_3^2 + 840 f_1 g_5 h_1^3 h_3^2 + 2520 f_4 g_1^4 h_1 h_2 h_3^2 + \\
& 15\,120 f_3 g_1^2 g_2 h_1 h_2 h_3^2 + 7560 f_2 g_2^2 h_1 h_2 h_3^2 + 10\,080 f_2 g_1 g_3 h_1 h_2 h_3^2 + 2520 f_1 g_4 h_1 h_2 h_3^2 + \\
& 280 f_3 g_1^3 h_3^3 + 840 f_2 g_1 g_2 h_3^3 + 280 f_1 g_3 h_3^3 + 126 f_6 g_1^6 h_1^5 h_4 + 1890 f_5 g_1^4 g_2 h_1^5 h_4 + \\
& 5670 f_4 g_1^2 g_2^2 h_1^5 h_4 + 1890 f_3 g_2^3 h_1^5 h_4 + 2520 f_4 g_1^3 g_3 h_1^5 h_4 + 7560 f_3 g_1 g_2 g_3 h_1^5 h_4 + \\
& 1260 f_2 g_3^2 h_1^5 h_4 + 1890 f_3 g_1^2 g_4 h_1^5 h_4 + 1890 f_2 g_2 g_4 h_1^5 h_4 + 756 f_2 g_1 g_5 h_1^5 h_4 + \\
& 126 f_1 g_6 h_1^5 h_4 + 1260 f_5 g_1^5 h_1^3 h_2 h_4 + 12\,600 f_4 g_1^3 g_2 h_1^3 h_2 h_4 + 18\,900 f_3 g_1 g_2^2 h_1^3 h_2 h_4 + \\
& 12\,600 f_3 g_1^2 g_3 h_1^3 h_2 h_4 + 12\,600 f_2 g_2 g_3 h_1^3 h_2 h_4 + 6300 f_2 g_1 g_4 h_1^3 h_2 h_4 + 1260 f_1 g_5 h_1^3 h_2 h_4 + \\
& 1890 f_4 g_1^4 h_1 h_2^2 h_4 + 11\,340 f_3 g_1^2 g_2 h_1 h_2^2 h_4 + 5670 f_2 g_2^2 h_1 h_2^2 h_4 + 7560 f_2 g_1 g_3 h_1 h_2^2 h_4 + \\
& 1890 f_1 g_4 h_1 h_2^2 h_4 + 1260 f_4 g_1^4 h_1^2 h_3 h_4 + 7560 f_3 g_1^2 g_2 h_1^2 h_3 h_4 + 3780 f_2 g_2^2 h_1^2 h_3 h_4 + \\
& 5040 f_2 g_1 g_3 h_1^2 h_3 h_4 + 1260 f_1 g_4 h_1^2 h_3 h_4 + 1260 f_3 g_1^3 h_2 h_3 h_4 + 3780 f_2 g_1 g_2 h_2 h_3 h_4 + \\
& 1260 f_1 g_3 h_2 h_3 h_4 + 315 f_3 g_1^3 h_1 h_4^2 + 945 f_2 g_1 g_2 h_1 h_4^2 + 315 f_1 g_3 h_1 h_4^2 + 126 f_5 g_1^5 h_1^4 h_5 + \\
& 1260 f_4 g_1^3 g_2 h_1^4 h_5 + 1890 f_3 g_1 g_2^2 h_1^4 h_5 + 1260 f_3 g_1^2 g_3 h_1^4 h_5 + 1260 f_2 g_2 g_3 h_1^4 h_5 + \\
& 630 f_2 g_1 g_4 h_1^4 h_5 + 126 f_1 g_5 h_1^4 h_5 + 756 f_4 g_1^4 h_1^2 h_2 h_5 + 4536 f_3 g_1^2 g_2 h_1^2 h_2 h_5 + \\
& 2268 f_2 g_2^2 h_1^2 h_2 h_5 + 3024 f_2 g_1 g_3 h_1^2 h_2 h_5 + 756 f_1 g_4 h_1^2 h_2 h_5 + 378 f_3 g_1^3 h_2^2 h_5 + \\
& 1134 f_2 g_1 g_2 h_2^2 h_5 + 378 f_1 g_3 h_2^2 h_5 + 504 f_3 g_1^3 h_1 h_3 h_5 + 1512 f_2 g_1 g_2 h_1 h_3 h_5 + \\
& 504 f_1 g_3 h_1 h_3 h_5 + 126 f_2 g_1^2 h_4 h_5 + 126 f_1 g_2 h_4 h_5 + 84 f_4 g_1^4 h_1^3 h_6 + 504 f_3 g_1^2 g_2 h_1^3 h_6 + \\
& 252 f_2 g_2^2 h_1^3 h_6 + 336 f_2 g_1 g_3 h_1^3 h_6 + 84 f_1 g_4 h_1^3 h_6 + 252 f_3 g_1^3 h_1 h_2 h_6 + 756 f_2 g_1 g_2 h_1 h_2 h_6 + \\
& 252 f_1 g_3 h_1 h_2 h_6 + 84 f_2 g_1^2 h_3 h_6 + 84 f_1 g_2 h_3 h_6 + 36 f_3 g_1^3 h_1^2 h_7 + 108 f_2 g_1 g_2 h_1^2 h_7 + \\
& 36 f_1 g_3 h_1^2 h_7 + 36 f_2 g_1^2 h_2 h_7 + 36 f_1 g_2 h_2 h_7 + 9 f_2 g_1^2 h_1 h_8 + 9 f_1 g_2 h_1 h_8 + f_1 g_1 h_9
\end{aligned}
$$